MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. Let A be 3×3 matrix such that $\det(A) = 5$. If $\det(3\,\text{adj}(2A\,\text{adj}(2A))) = 2^{\alpha}\cdot 3^{\beta}\cdot 5^{\gamma}$, then $(\alpha + \beta + \gamma)$ is equal to

(1) 25 (2) 26

(3) 27 (4) 28

Answer (3)

Sol. $|3\text{adj}\,(2A\,\text{adj}(2A))| = 3^3|\,2A(\text{adj}\,2A)|^2$

$= 3^3(2^3)^2|A|^2|\text{adj}(2A)|^2$

$= 3^3\cdot 2^6\cdot 5^2(|(2A)|^2)^2$

$= 3^3\cdot 2^6\cdot 5^2\cdot\ |2A|^4$

$= 3^3\cdot 2^6\cdot 5^2\cdot (2^3)^4\cdot |A|^4$

$= 2^{18}\cdot 3^3\cdot 5^6 = 2^{\alpha}\cdot 3^{\beta}\cdot 5^{\gamma}$

$\Rightarrow\ \alpha = 18$

$\beta = 3$

$\gamma = 6$

$\Rightarrow\ \alpha + \beta + \gamma = 27$

2. The sum of all rational numbers in $\left(2+\sqrt{3}\right)^8$ is

(1) 18117 (2) 18817

(3) 17280 (4) 1800

Answer (2)

Sol. $S = {}^8C_0(2)^8 + {}^8C_1 2^7\left(\sqrt{3}\right) + \ldots + {}^8C_8\left(\sqrt{3}\right)^8$

Sum of rational terms

$= {}^8C_0(2)^8 + {}^8C_2 2^6\left(\sqrt{3}\right)^2 + {}^8C_4(2)^4\left(\sqrt{3}\right)^4 + {}^8C_6(2)^2\left(\sqrt{3}\right)^2 + {}^8C_8\left(\sqrt{3}\right)^8$

= 18,817

3. If the sum $\sum_{r=1}^{9}\left(\frac{r+3}{2^r}\right)\cdot{}^9C_r = \alpha\cdot\left(\frac{3}{2}\right)^9 - \beta$, then the value of $(\alpha + \beta)^2$ is equal to

(1) 9 (2) 81

(3) 27 (4) 36

Answer (2)

Sol. $\sum_{r=1}^{9}\left(\frac{r+3}{2^r}\right)\cdot{}^9C_r = \sum_{r=1}^{9}\frac{r}{2^r}\cdot\frac{9}{r}\cdot{}^8C_{r-1} + \sum_{r=1}^{9}3\cdot{}^9C_r\left(\frac{1}{2}\right)^r$

$= \frac{9}{2}\sum_{r=1}^{9}{}^8C_{r-1}\left(\frac{1}{2}\right)^{r-1} + 3\sum_{r=1}^{9}{}^9C_r\left(\frac{1}{2}\right)^r$

$= \frac{9}{2}\sum_{r=0}^{8}{}^8C_{r-1}\left(\frac{1}{2}\right)^r + 3\sum_{r=1}^{9}{}^9C_r\left(\frac{1}{2}\right)^r$

$= \frac{9}{2}\left(1+\frac{1}{2}\right)^8 + 3\left[\left(1+\frac{1}{2}\right)^9 - {}^9C_0\left(\frac{1}{2}\right)^0\right]$

$= \frac{9}{2}\cdot\frac{3^8}{2^8} + 3\left[\frac{3^9}{2^9} - 1\right]$

$= \frac{3^{10}}{2^9} + \frac{3^{10}}{2^9} - 3 = 4\cdot\frac{3^{10}}{2^{10}} - 3$

$= 4\left(\frac{3}{2}\right)^{10} - 3$

$= 6\left(\frac{3}{2}\right)^9 - 3$

$\alpha = 6,\ \beta = 3 \Rightarrow (\alpha + \beta)^2 = 81$

4. Let $S_n = 1 + 3 + 11 + 25 + 45 + \ldots$ Then sum upto 20^{th} term equals

(1) 6200 (2) 7200

(3) 7240 (4) 6240

Answer (3)

Sol.
$$S = 1 + 3 + 11 + 25 + ... + T_n$$
$$S = 1 + 3 + 11 + ...T_{n-1} + T_n$$
$$T_n = 1 + 2 + 8 + 14 + ... + (T_n - T_{n-1})$$

$$T_n = 1 + \frac{n-1}{2}[4 + (n-2)6]$$

$$= 1 + \left(\frac{n-1}{2}\right)[6n - 8]$$

$= 1 + (n - 1)(3n - 4)$

$= 1 + 3n^2 - 4n - 3n + 4$

$T_n = 3n^2 - 7n + 5$

$$S_n = \sum Tn = 3\sum n^2 - 7\sum n + \sum 5$$

$$= \frac{3n(n+1)(2n+1)}{6} - \frac{7n(n+2)}{2} + 5n$$

now $n = 20$

$$= \frac{3 \times 20 \times 21 \times 41}{6} - \frac{7 \times 20 \times 21}{2} + 5 \times 20 = 7240$$

5. Evaluate $\int x^3 \sqrt{1 - x^2}\, dx$

(1) $\frac{-1}{15}(1 - x^2)^{3/2}(3x^2 + 2) + C$

(2) $\frac{1}{3}(1 + x^2)^{2/3} - \sqrt{1 - x^2} + C$

(3) $\frac{2}{3}(1 - x^2)^{3/2}(3x^2 + 2) + C$

(4) $\frac{1}{3}(1 - x^2)^{2/3} + \sqrt{1 - x^2} + C$

Answer (1)

Sol. $\int x^3 \sqrt{1 - x^2}\, dx$

Put $1 - x^2 = t^2$

$-2x\, dx = 2t dt$

$$-\int t^2(1 - t^2)dt = -\left[\frac{t^3}{3} - \frac{t^5}{5}\right] + C$$

$$\frac{t^5}{5} - \frac{t^3}{3} + C$$

$$= \frac{(1 - x^2)^{5/2}}{5} - \frac{(1 - x^2)^{3/2}}{3} + C$$

$$= \frac{(1 - x^2)^{3/2}}{15}\left[3(1 - x^2) - 5\right] + C$$

$$= \frac{-(1 - x^2)^{3/2}}{15}(3x^2 + 2) + C$$

6. A relation $R = \{(x, y); x, y \in A = \{-3, -2, -1, 0, 1, 2, 3\}$ such that $x^2 + 2y \le 4\}$. If the number of ordered pairs in relation R be r and number of ordered pairs required to add in R so that it becomes reflexive relations is m, then $r + m$ is equal to

(1) 26 (2) 28

(3) 24 (4) 23

Answer (2)

Sol. $x^2 + 2y \le 4$

$A = \{-3, -2, -1, 0, 1, 2, 3\}$

$x^2 \le 4 - 2y$

For $y = -3$

$x^2 \le 4 - (2(-3))$

$x^2 \le 10$

$\Rightarrow x^2 \in \{-3, -2, -1, 0, 1, 2, 3\}$

For $y = -2$

$x^2 \le 4 - 2(-2)$

$x^2 \le 8$

$\Rightarrow x \in \{-2, -1, 0, 1, 2\}$

For $y = -1$

$x^2 \le 4 - (2(-1))$

$x^2 \le 6$

$\Rightarrow x \in \{-2, -1, 0, 1, 2\}$

For $y = 0$

$x^2 \le 4$

$\Rightarrow x \in \{-2, -1, 0, 1, 2\}$

For $y = 1$

$x^2 \le 2$

$\Rightarrow\ x \in \{-1, 0, 1\}$

For $y = 2$

$x^2 \le 0$

$\Rightarrow\ x \in \{0\}$

For $y = 3$

$x^2 \le -2 \Rightarrow$ No value of x

Tota number of ordered pair in relation R is, $r = 26$

For it to be reflexive we have to add {(3, 3), (2, 2)}

$\Rightarrow\ m = 2$

$\Rightarrow\ r + m = 28$

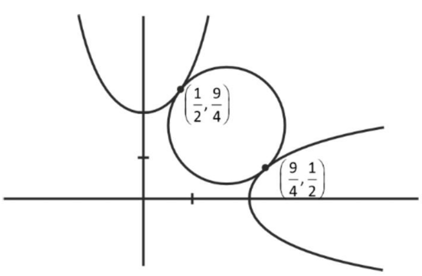
7. The radius of circle touching both parabolas $y = x^2 + 2$ and $x = y^2 + 2$ is

(1) $\frac{7\sqrt{2}}{2}$ (2) $\frac{7\sqrt{2}}{6}$

(3) $\frac{7\sqrt{2}}{8}$ (4) $\frac{7\sqrt{2}}{4}$

Answer (4)

Sol.



The circle will have its centre at $x = y$ line and since these parabolas are symmetric about the line $y = x$. The slope will be of tangents at closest points.

$\Rightarrow\ y^2 = x - 2 \qquad \Rightarrow\ 2y\frac{dy}{dx} = 1 \quad \Rightarrow\ y = \frac{1}{2}$

$\Rightarrow$ Point will be $\left(\frac{9}{4}, \frac{1}{2}\right)$

Similarly on $x^2 = y - 2$

$\Rightarrow\ 2x = \frac{dy}{dx} = 1 \qquad \Rightarrow\ x = \frac{1}{2}$

$\Rightarrow\ \left(\frac{1}{2}, \frac{9}{4}\right)$

Circle's diameter will be equal to shortest distance

$$2r = \sqrt{\left(\frac{9}{4} - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{9}{4}\right)^2} = \sqrt{\frac{7}{4} \times 2} = \sqrt{\frac{7}{2}}$$

$$\Rightarrow\ r = \frac{1}{2}\sqrt{\frac{7}{2}} = \sqrt{\frac{7}{8}} = \frac{7\sqrt{2}}{4}$$

8. Let $3x + 2\tan x = \pi$, $x \in [-2\pi, 2\pi] - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\right\}$

Then number of values of x satisfying the above condition is

(1) 4

(2) 5

(3) 6

(4) 7

Answer (2)

Sol. $3x + 2\tan x = \pi$

$2\tan x = \pi - 3x$

$\tan x = \frac{\pi - 3x}{2}$

5 solution

$\because$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ well get 1 solution as $\tan x$ is increasing in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \left.\begin{array}{l} \left(\frac{-3\pi}{2}, \frac{\pi}{2}\right) \to 1 \text{ solution} \\ \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to 1 \text{ solution} \\ \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \to 1 \text{ solution} \\ \left(\frac{3\pi}{2}, 2\pi\right) \to 1 \text{ solution} \\ \left(-2\pi, \frac{-3\pi}{2}\right) \to 1 \text{ solution} \end{array}\right\} 5$$

9. Let $\int_0^x g(t)dt = x - \int_0^x tg(t)dt,\ x \geq 0$ and $\frac{dy}{dx} - y\tan x = 2(x+1)\sec x g(x)$ satisfying the condition $y(0) = 0$. Then $y\left(\frac{\pi}{3}\right)$ is

(1) $\frac{2\pi}{3}$ (2) $\frac{4\pi}{3}$

(3) π (4) 2π

Answer (2)

Sol. Differentiate both side w.r.t x

$g(x) = 1 - xg(x)$

$g(x)(1 + x) = 1$

$g(x) = \frac{1}{1+x}$

Also $\frac{dy}{dx} - y\tan x = 2(x+1)\sec x g(x)$

$I.F = e^{-\int \tan x dn.}$

$I.F = e^{-(-\ln\cos x)}$

$I.F = \cos x$

$y\cos x = \int 2(x+1)\sec x \frac{1}{(1+x)}\cos x dx$

$y\cos x = \int 2dx$

$y\cos x = 2x + c$ ...(i)

$y(0) = 0$

$\Rightarrow c = 0$

from (i)

$y\cos x = dx$

Put $x = \frac{\pi}{3}$

$y \cdot \frac{1}{2} = \frac{2\pi}{3}$

$y = \frac{4\pi}{3}$

10. If $f(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x + 1 \\ 27 & 28 & 27 \\ 1 & 1 & 1 \end{vmatrix}$. Then, the value of $f'(x) + f(x)$ is

(1) -1

(2) 28

(3) 27

(4) 1

Answer (1)

Sol. $f(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x + 1 \\ 27 & 28 & 27 \\ 1 & 1 & 1 \end{vmatrix}$

$f(x) = \sin x(1) - \cos x(0) + (\sin x + \cos x + 1)(-1)$

$f(x) = \sin x - \sin x - \cos x - 1$

$f(x) = -\cos x - 1$

$f'(x) = \sin x$

$f''(x) - \cos x$

$f(x) + f''(x) = -\cos x - 1 + \cos x$

$= -1$

11. Let α, β are the roots of the equation $x^2 + \sqrt{3}x - 16 = 0$ and γ, δ are the roots of the equation $x^2 + 3x - 1 = 0$. If $Q_n = \alpha^n + \beta^n\ \forall\ n \in N$ and $P_n = \gamma^n + \delta^n\ \forall\ n \in N$ then the value of $\frac{Q_{25} + \sqrt{3}Q_{24}}{2Q_{23}} + \left(\frac{P_{25} - P_{23}}{P_{24}}\right)$ is

(1) 5 (2) 6

(3) 7 (4) 8

Answer (1)

Sol. $x^2 + 3x - 1 = 0 \begin{smallmatrix}\nearrow \gamma \\ \searrow \delta\end{smallmatrix} \Rightarrow x^2 - 1 = -3x$

$\Rightarrow P^n = \gamma^n + \delta^n$

$P_{25} - P_{23} = (\gamma^{25} - \gamma^{23}) + (\delta^{25} - \delta^{23})$

$= \gamma^{23}(\gamma^2 - 1) + \delta^{23}(\delta^2 - 1)$

$= \gamma^{23}(-3\gamma) + \delta^{23}(-3\delta)$

$= -3[\gamma^{24} + \delta^{24}]$

$$\Rightarrow \frac{P_{25} - P_{23}}{P_{24}} = (-3)$$

Similarly

$$x^2 + \sqrt{3}x - 16 = 0 \; \begin{matrix} \alpha \\ \beta \end{matrix} \quad Q_n = \alpha^n + \beta^n$$

$$\Rightarrow Q_{25} + \sqrt{3}Q_{24} = (\alpha^{25} + \sqrt{3}\alpha^{24}) + (\beta^{25} + \sqrt{3}\beta^{24})$$

$$= \alpha^{23}(\alpha^2 + \sqrt{3}\alpha) + \beta^{23}(\beta^2 + \sqrt{3}\beta)$$

$$= \alpha^{23}(16) + 16\beta^{23}$$

$$\Rightarrow \frac{Q_{25} + \sqrt{3}Q_{24}}{2 \cdot Q_{23}} = \frac{16(\alpha^{23} + \beta^{23})}{2(\alpha^{23} + \beta^{23})} = 8$$

$$\Rightarrow \frac{Q_{25} + \sqrt{3}Q_{24}}{2Q_{23}} + \frac{(P_{25} - P_{23})}{P_{24}} = 8 + (-3) = 5$$

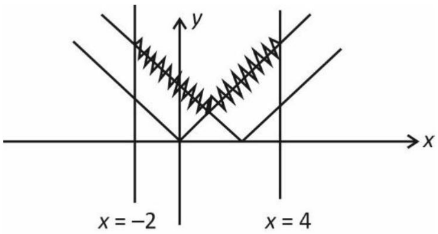
12. If $y = \max\{|x|, x, |x-2|\}$, then the area under the curve from $x = -2$ to $x = 4$ is (in sq. units)

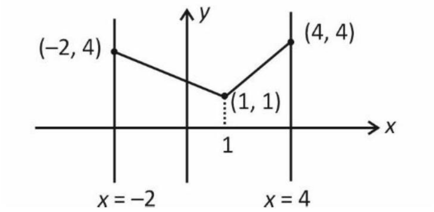
(1) 15 (2) 20

(3) 12 (4) 8

Answer (1)

Sol. $\max\{|x|, x, |x-2|\}$





$$\text{Area} = \frac{1}{2}[(1+4)\times 3] + \frac{1}{2}[1+4]\times 3$$

$$= \frac{15}{2} + \frac{15}{2} = 15 \text{ sq. unit}$$

13. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$. Let $\vec{c}$ is a unit vector such that $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$. If $\vec{c} = \lambda\hat{i} + \mu\hat{j}$ and $\vec{d}$ is a vector perpendicular to $\vec{c}$ and $\vec{a}$, then $|\lambda\vec{c} + \mu\vec{d}|^2$ is equal to

(1) $\frac{6}{25}$

(2) $\frac{61}{25}$

(3) $\frac{41}{25}$

(4) $\frac{36}{25}$

Answer (2)

Sol. $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$

$$\Rightarrow (\vec{a} - \vec{b}) \times \vec{c} = 0$$

$$\Rightarrow \vec{c} \,||\, (\vec{a} - \vec{b}) = (-\hat{i} + 2\hat{j})$$

$$\Rightarrow \vec{c} = \frac{-i}{\sqrt{5}} + \frac{2j}{\sqrt{5}} = \lambda\hat{i} + \mu\hat{j}$$

$$\Rightarrow \lambda = \frac{-1}{\sqrt{5}}, \mu = \frac{2}{\sqrt{5}}$$

$$\vec{c} \cdot \vec{d} = 0$$

$$|\lambda\vec{c} + \mu\vec{d}|^2 = \lambda^2|\vec{c}|^2 + 2\lambda\mu(0) + \mu^2|\vec{d}|^2$$

$$\vec{d} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{vmatrix} = \hat{i}\left(\frac{-2}{\sqrt{5}}\right) + \left(\frac{-1}{\sqrt{5}}\right)\hat{j} + \left(\frac{3}{\sqrt{5}}\right)\hat{k}$$

$$|\vec{d}| = \sqrt{\frac{4}{5} + \frac{1}{5} + \frac{9}{5}} = \sqrt{\frac{14}{5}}$$

$$\left(\frac{1}{5}\right)\times(1) + 0 + \left(\frac{4}{5}\right)\cdot\left(\frac{14}{5}\right) = \frac{5}{25} + \frac{56}{25} = \left(\frac{61}{25}\right)$$

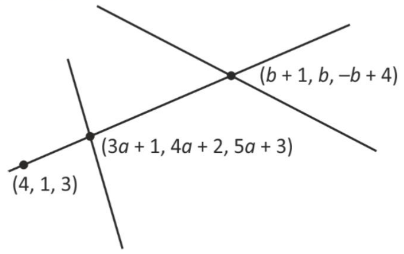
14. Let a line passing through (4, 1, 3) intersects the line $l_1 : \frac{x-1}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ at (α, β, γ) and $l_2 : x - 1 = y = -z + 4$ at (a, b, c) then $\begin{vmatrix} 63 & 21 & -21 \\ \alpha & \beta & \gamma \\ a & b & c \end{vmatrix}$ is equal to

(1) 102 (2) 204

(3) 63 (4) 21

Answer (2)

Sol.



$$\frac{3a+1-4}{b+1-4} = \frac{4a+2-1}{b-1} = \frac{5a+3-3}{-b+4-3}$$

$$\frac{3a-3}{b-3} = \frac{4a+1}{b-1} = \frac{-5a}{b-1}$$

$$\Rightarrow 4a+1 = -5a \Rightarrow a = \frac{-1}{9}$$

$$\Rightarrow \frac{3\left(\frac{-1}{9}-1\right)}{b-3} = \frac{4\left(-\frac{1}{9}\right)+1}{b-1} = \frac{\frac{-10}{3}}{b-3} = \frac{\frac{5}{9}}{b-1}$$

$$\Rightarrow b = \frac{9}{7}$$

$$\Rightarrow l_1(\alpha, \beta, \gamma) = \left(\frac{2}{3}, \frac{14}{7}, \frac{22}{9}\right) \equiv \left(\frac{6}{9}, \frac{14}{9}, \frac{22}{9}\right)$$

$$l_2\,(a, b, c) = \left(\frac{16}{7}, \frac{9}{7}, \frac{19}{7}\right)$$

$$\Rightarrow \begin{vmatrix} 63 & 21 & -21 \\ \frac{6}{9} & \frac{14}{9} & \frac{22}{9} \\ \frac{16}{7} & \frac{9}{7} & \frac{19}{7} \end{vmatrix} = 204$$

15. Let a_1, a_2, a_3....be the terms of an increasing G.P. such that $a_3 \cdot a_5 = 729$ and $a_3 + a_5 = \frac{111}{4}$, then $24(a_1 + a_2 + a_3)$ is equal to

(1) 139 (2) 129

(3) 125 (4) 119

Answer (2)

Sol. Let $a_3 + a_5 = \frac{111}{4}$

and $a_3 \cdot a_5 = 729$

$\Rightarrow (ar^2)\cdot(ar^4) = (27)^2$

$\Rightarrow ar^3 = 27, a_i > 0$

$a_4 = 27$

$$\Rightarrow a_3 = \frac{27}{r}; a_5 = 27r$$

$$27r + \frac{27}{r} = \frac{111}{4} = \frac{37\times 3}{4}$$

$$\Rightarrow r + \frac{1}{r} = \frac{37}{36} \Rightarrow r \Rightarrow \frac{1}{6}, 6 \Rightarrow r = 6$$

$$24(a_1 + a_2 + a_3) = 24\left[\frac{27}{216} + \frac{27}{36} + \frac{27}{6}\right]$$

$$= 24\left[\frac{1}{8} + \frac{3}{4} + \frac{9}{2}\right]$$

$= 3 + 18 + 108 = 129$

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 5 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21.

22.

23.

24.

25.